INDIAN STATISTICAL INSTITUTE
Back-paper Exam
Algebra-I
2018-2019

Total marks: 100
Time: 3 hours

Answer all questions.

1. (a) Find the number of elements of order 10 in $G = \mathbb{Z}_{100} \times \mathbb{Z}_{25}$.
   (b) Hence find the number of cyclic subgroups of order 10 in G. (10+5)

2. (a) Let G be a group and H be a subgroup of G. Let G act on the set X_H of all left cosets of H in G. Determine the kernel of this action.
   (b) Prove that if H is of finite index n in G then there is a normal subgroup K of G with $K \subseteq H$ and $[G : K] \leq n!$. (5+10)

3. Prove that if p is prime dividing the order of a group G, then there exists an element of order p in G. (10+10)

4. Prove that the automorphism group of the cyclic group of order n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{\times}$. (15)

5. (a) Define $Inn(G)$, the group of inner automorphisms of a group G. Show that
$$G/Z(G) \cong Inn(G).$$
   (b) Compute $Inn(Q_8)$. (10+10)

6. Show that a group of order p^2q, where p and q are distinct primes, has a normal Sylow subgroup for either p or q. (15)

**** End ****